# Chapter 2 Homework 2012

Saturday, August 18, 2012
8:00 AM

1 a $S_0(t) = 1 - F_0(t) = \left(1 - \frac{t}{125}\right)^{1/5}$

b $Pr[T_0 \le t] = F_0(t) =$
$1 - \left(1 - \frac{t}{125}\right)^{1/5}$

c $Pr[T_0 > t] = S_0(t) = \left(1 - \frac{t}{125}\right)^{1/5}$

d $S_x(t) = \frac{S_0(x+t)}{S_0(x)}$

$= \frac{\left(1 - \frac{x+t}{125}\right)^{1/5}}{\left(1 - \frac{x}{125}\right)^{1/5}}$

$= \left(\frac{125 - x - t}{125 - x}\right)^{1/5}$

e $S_0(25) = \left(1 - \frac{25}{125}\right)^{1/5} = 0.95635$

f $S_{25}(50) = \frac{S_0(75)}{S_0(25)} =$

$\left(\frac{125 - 25 - 50}{125 - 25}\right)^{1/5} = 0.87055$

g $Prob\,[25 \le T_{25} \le 50]$

$= S_{25}(25) - S_{25}(50)$

$= \left(\frac{125 - 25 - 25}{125 - 25}\right)^{1/5} - \left(\frac{125 - 25 - 50}{125 - 25}\right)^{1/5}$

$= 0.07354$

(h) $\omega$ is the smallest value of $t$ such that $S_0(t) = 0$

$$\left(1 - \frac{t}{125}\right)^{1/5} = 0$$

$$1 - t/125 = 0$$

$$t = 125 \Rightarrow \omega = 125$$

(i) $$\mu_x = -\frac{1}{S_0(x)} \frac{d}{dx} S_0(x)$$

$$-\left(1 - \frac{x}{125}\right)^{-\frac{1}{5}} \left(\frac{1}{5}\right)\left(1 - \frac{x}{125}\right)^{-4/5}\left(\frac{-1}{125}\right)$$

$$= \frac{1}{625 - 5x}$$

(j) $$\mu_{25} = \frac{1}{625 - 5(25)} = 0.002$$

(k) $$\mu_{100} = \frac{1}{625 - 5(100)} = 0.008$$

(l) $${}_t p_x = S_x(t) = \left(\frac{125 - x - t}{125 - x}\right)^{1/5}$$

(m) $${}_{10} p_{50} = \left(\frac{125 - 50 - 10}{125 - 50}\right)^{1/5} =$$

$$0.97179$$

(n) $${}_t q_x = 1 - {}_t p_x = \left(\frac{125 - x - t}{\phantom{x}}\right)^{1/5}$$

$$1 - \left( \overline{\phantom{xx}} \; 125 - x \right)$$

(o) $_{10}q_{50} = 1 - {}_{10}p_{50} = 1 - 0.97179$

$$= 0.02821$$

(p) $_{10}p_{50} + {}_{10}q_{50} = 1$

(q) $p_{50} = \left( \frac{125-50-1}{125-50} \right)^{1/5} = 0.99732$

(r) $_{u|t}q_x = S_x(u) - S_x(u+t)$

$$= \frac{S_0(x+u)}{S_0(x)} - \frac{S_0(x+u+t)}{S_0(x)}$$

$$= \frac{(125-x-u)^{1/5} - (125-x-u-t)^{1/5}}{(125-x)^{1/5}}$$

(s) $f_x(t) = -\frac{d}{dt} S_x(t)$

$$= -\frac{d}{dt} \left( \frac{125-x-t}{125-x} \right)^{1/5}$$

$$= -\tfrac{1}{5} \left( \frac{125-x-t}{125-x} \right)^{-4/5} \left( -\frac{1}{125-x} \right)$$

$$= \frac{1}{625-5x} \left( \frac{125-x-t}{125-x} \right)^{-4/5}$$

or $f_x(t) = {}_tp_x \, \mu_{x+t} =$

$$\left( \frac{125-x-t}{125-x} \right)^{1/5} \frac{1}{625-5(x+t)}$$

$$\left(\frac{125-x-t}{125-x}\right)^{1/5} \frac{1}{625-5(x+t)}$$

$$= \left(125-x-t\right)^{1/5}\left(125-x\right)^{-1/5}$$

$$\left(\frac{1}{5}\right)\left(\frac{1}{125-x-t}\right)$$

$$= \left(125-x-t\right)^{-4/5}\left(\frac{1}{5}\right)$$

$$\left(125-x\right)^{-1}\left(125-x\right)^{+4/5}$$

$$= \left(\frac{125-x-t}{125-x}\right)^{-4/5}\left(\frac{1}{625-5x}\right)$$

(t) $E[T_x] = \int_0^\infty t\,p_x\,dt$

$$= \int_0^{125-x}\left(1-\frac{t}{125-x}\right)^{1/5} dt$$

$$\boxed{\begin{aligned} y = 1-\frac{t}{125-x} \ni t &= (1-y)(125-x) \\ dt &= -dy\,(125-x) \end{aligned}}$$

$$= \int_1^0 y^{1/5}\,(-dy)(125-x)$$

$$= 125-x\int_0^1 y^{1/5}\,dy$$

$$= 5/6\,(125-x)$$

(u) $\mathring{e}_x = E[T_x] = 5/6\,(125-x)$

(v) $Var[T_x] = E[(T_x)^2] - (\mathring{e}_x)^2$

$$E[(T_x)^2] = 2\int_0^{125-x} t \cdot {}_tp_x \, dt$$

$$= 2\int_0^{125-x} t\left(1 - \frac{t}{125-x}\right) dt$$

$$= 2\int_1^0 (1-y)(125-x)(y)^{1/5}(-dy)(125-x)$$

$$= 2\,(125-x)^2 \int_0^1 \left(y^{1/5} - y^{6/5}\right) dy$$

$$= 2\,(125-x)^2\left(5/6 - 5/11\right)$$

$$= \frac{25}{33}(125-x)^2$$

$$Var[T_x] = \frac{25}{33}(125-x)^2 - \left[5/6\,(125-x)\right]^2$$

$$= \frac{25}{396}(125-x)^2$$

(w) $\sqrt{Var[T_{50}]} =$

$$\sqrt{\frac{25}{396}(125-50)^2} = 18.84446$$

(x) $E[K_{120}] = {}_1p_{120} + {}_2p_{120} +$

$${}_3p_{120} + {}_4p_{120}$$

$$= \left(\frac{125-121}{125-120}\right)^{\frac{1}{5}} + \left(\frac{3}{5}\right)^{1/5} + \left(\frac{2}{5}\right)^{1/5} + \left(1/5\right)^{1/5}$$

$$= 3.41656$$

(g) $Var[K_{120}] = E[K_{120}^2] - (e_x)^2$

$$E[K_{120}^2] = 2\sum_{K=1}^{\infty} k \cdot {}_kp_x - e_x$$

$$= 2\left[(1)(4/5)^{1/5} + (2)(3/5)^{1/5} + (3)(2/5)^{1/5} + (4)(1/5)^{1/5}\right] - 3.41656$$

$$= 12.90122$$

$$Var[K_{120}] = 12.90122 - (3.41656)^2$$

$$= 1.22830$$

(2) (a)

$$\mu_x = Bc^x = (0.00027)(1.1)^x$$

(b)

$$\mu_{25} = (0.00027)(1.1)^{25} =$$

$$0.0029254$$

(c)

$$\mu \quad (0.00027)(1.1)^{100}$$

(c)

$$\mu_{100} = (0.00027)(1.1)^{100}$$
$$= 3.72077$$

(d)

$$S_0(t) = \exp\left[\frac{-B}{\ln c} c^x (c^t - 1)\right]$$
$$= \exp\left[\frac{-0.00027}{\ln 1.1} (1.1)^0 (c^t - 1)\right]$$
$$= \exp\left[\frac{-0.00027}{\ln (1.1)} (1.1^t - 1)\right]$$

(e) $\Pr[T_0 \le t] = 1 - S_0(t)$

$$= 1 - \exp\left[\frac{-0.00027}{\ln (1.1)} (1.1^t - 1)\right]$$

(f) $\Pr[T_0 > t] = S_0(t)$

$$= \exp\left[\frac{-0.00027}{\ln (1.1)} (1.1^t - 1)\right]$$

(g) $S_x(t) = \exp\left[\frac{-B}{\ln c} c^x (c^t - 1)\right]$

$$= \exp\left[\frac{-0.00027}{\ln (1.1)} (1.1)^x (1.1^t - 1)\right]$$

(h) $S_0(25) = \exp\left\{\frac{-0.00027}{\ln 1.1} (1.1^{25} - 1)\right\}$

$$= 0.97252$$

25, 10

$= 0.41252$

(i) $S_{25}(50) = \exp\left[\frac{-0.00027}{\ln 1.1}(1.1)^{25}(1.1^{50}-1)\right]$

$= 0.028088$

(j) $S_{25}(25) - S_{25}(50)$

$= \exp\left[\frac{-0.00027}{\ln 1.1}(1.1)^{25}(1.1^{25}-1)\right]$

$- \exp\left[\frac{-0.00027}{\ln 1.1}(1.1)^{25}(1.1^{50}-1)\right]$

$= 0.73944 - 0.02809$
$= 0.71135$

(k) $\omega = \infty$ with Gompertz there is no point at which ${}_tp_0 = 0$

(l) ${}_tp_x = S_x(t) =$

$\exp\left[\frac{-0.00027}{\ln 1.1}(1.1)^{x}(1.1^{t}-1)\right]$

(m) ${}_{10}p_{50} = \exp\left[\frac{-0.00027}{\ln 1.1}(1.1)^{50}(1.1^{10}-1)\right]$

$= 0.58860$

(n) ${}_tq_x = 1 - {}_tp_x =$

$1 - \exp\left[\frac{-0.00027}{\ln 1.1}(1.1)^{x}(1.1^{t}-1)\right]$

(o) ${}_{10}q_{50} = 1 - {}_{10}p_{50} = 1 - 0.58860$

(o) $_{10}q_{50} = 1 - {}_{10}p_{50} = 1 - 0.58860$
$= 0.41140$

(p) $_{10}p_{50} + {}_{10}q_{50} = 1$

(q) $p_{50} = \exp\left[\frac{-0.00027}{\ln 1.1}(1.1)^{50}(1.1-1)\right]$
$= 0.96729$

(r) $_{u|t}q_x = {}_u p_x - {}_{u+t}p_x$
$= \exp\left[\frac{-0.00027}{\ln 1.1}(1.1)^{x}(1.1^{u}-1)\right]$
$- \exp\left[\frac{-0.00027}{\ln 1.1}(1.1)^{x}(1.1^{u+t}-1)\right]$

(s) $f_x(t) = {}_t p_x \, \mu_{x+t}$
$= \exp\left[\frac{-0.00027}{\ln 1.1}(1.1)^{x}(1.1^{t}-1)\right] \cdot$
$(0.00027)(1.1^{x+t})$

(3) (a) $_t p_x = e^{-\int_0^t \mu_{x+t}\, dt}$
$= e^{-\int_0^t c \cdot dt} = e^{-tc}$

(b) $_t q_x = 1 - {}_t p_x = 1 - e^{-tc}$

(c) $\omega$ is $\infty$ since $e^{-tc} \neq 0$
for any value of $t$

(d) $\mathring{e}_x = \int^{\infty} {}_t p_x \, dt =$

$$\int_0^\infty e^{-tc}\,dt = -\frac{e^{-tc}}{c}\Big|_0^\infty$$

$$= 0 - -\frac{1}{c} = \frac{1}{c}$$

(e) $Var = E(T_x^2) - (\mathring{e}_x)^2$

$$E(T_x^2) = \int_0^\infty t \; {}_tp_x \, dt$$

$$= \int_0^\infty t\, e^{-tc}\,dt \qquad \begin{array}{ll} u = t & dv = e^{-tc} \\ du = 1 & v = \frac{e^{-tc}}{-c} \end{array}$$

$$= -\frac{t\,e^{-tc}}{c}\Big|_0^\infty - \int_0^\infty \frac{e^{-tc}}{-c}\,dt$$

$$= 0 - 0 + \frac{e^{-tc}}{-c^2}\Big|_0^\infty = \frac{1}{c^2}$$

(f) $e_x = {}_1p_x + {}_2p_x + {}_3p_x + \cdots$

$$= e^{-c} + e^{-2c} + e^{-3c} + \cdots$$

$$= \frac{e^{-c} - 0}{1 - e^{-c}} = \frac{1}{e^c - 1}$$

(g) $e^{-10c}$

(h) $e^{-10c}$

i $e^{-10c}$

(j) It is not a reasonable model because mortality is not a function of age. A ...

function of age. A person age 500 has the same probability of living 10 years as a person age 10.

(4) (a) $F_0(x) = {}_x q_0 = \frac{x^2}{10{,}000}$

(b) $S_0(x) = 1 - F_0(x)$

$= 1 - \frac{x^2}{10{,}000}$

(c) $S_x(t) = \frac{S_0(x+t)}{S_0(x)} =$

$\frac{1 - \frac{(x+t)^2}{10000}}{1 - \frac{x^2}{10{,}000}} = \frac{10{,}000 - (x+t)^2}{10{,}000 - x^2}$

(d) $f_0(x) = \frac{d}{dx} F_0(x) = \frac{d}{dx} \frac{x^2}{10{,}000}$

$= \frac{2x}{10{,}000} = \frac{x}{5000}$

(e) $E[T_x] = \int_0^{\infty} {}_t p_x \, dt$

$= \int_0^{100-x} \frac{10000 - x^2 - 2xt - t^2}{10000 - x^2} dt$

$= \left. \frac{(10000 - x^2)(t) - xt^2 - \frac{t^3}{3}}{10000 - x^2} \right|_0^{100-x}$

$= \frac{(100-x)(100+x)(100-x) - x(100-x)^2 - \frac{(100-x)^3}{3}}{(100-x)(100+x)}$

$$= \frac{\quad}{(100-x)(100+x)}$$

$$= \frac{(100+x)(100-x) - x(100-x) - \frac{(100-x)^2}{3}}{100+x}$$

$$= \frac{(100-x)\left(100+x-x-\left(\frac{100-x}{3}\right)\right)}{100+x}$$

$$= \frac{(100-x)\left(100-\frac{100}{3}+\frac{x}{3}\right)}{100+x}$$

$$= \frac{(100-x)(200+x)}{300+3x}$$

$$E[T_0] = \frac{(100)(200)}{300} = 66.\overline{66}$$

(f)

$$Var[T_0] = E(T_0^2) - (E[T_0])^2$$

$$E[T_0^2) = 2\int_0^{100} t \ {}_t p_0 \ dt$$

$$= 2\int_0^{100} t \cdot \left(\frac{10000-t^2}{10000}\right) dt$$

$$= \left.\frac{5000t^2 - \frac{t^4}{4}}{5000}\right|_0^{100}$$

$$= 5000$$

$$Var = 5000 - (66.\overline{66})^2$$

$= 555.56$

(g) $_{40}q_0 = \frac{40^2}{10000} = 0.16$

(h) $_{40}p_0 = 1 - {}_{40}q_0 = 1 - 0.16 = 0.84$

(i) $\Pr(40 < T_0 < 60) = {}_{40}p_0 - {}_{60}p_0$

$= 0.84 - \left[1 - \frac{60^2}{10000}\right] = 0.20$

(j) $\mu_x = \frac{-\frac{d}{dx} S_0(x)}{S_0(x)}$

$$= \frac{-\frac{d}{dx}\left(1 - \frac{x^2}{10{,}000}\right)}{1 - \frac{x^2}{10{,}000}}$$

$$= \frac{\frac{x}{5000}}{1 - \frac{x^2}{10{,}000}}$$

$$= \frac{2x}{10{,}000 - x^2}$$

(k) $\mu_{75} = \frac{2(75)}{10{,}000 - (75)^2} = \frac{6}{175}$

(l) $_tp_x = S_x(t) = \frac{10{,}000 - (x+t)^2}{10000 - x^2}$

(m) $_tq_x = 1 - {}_tp_x =$

$$1 - \frac{10000 - (x+t)^2}{10000 - x^2} =$$

$$\frac{10,000 - x^2 - 10000 + x^2 + 2xt + t^2}{10,000 - x^2}$$

$$= \frac{2xt + t^2}{10,000 - x^2}$$

(n) $${}_tq_{75} = \frac{150t + t^2}{4375}$$

(o) $${}_tp_{75} = 1 - {}_tq_{75}$$

$$= \frac{4375 - 150t - t^2}{4375}$$

(p) $$E[T_x] = \frac{(100-x)(200+x)}{300+3x}$$

From Part (e)

(q) $$E[T_{75}] = \frac{(100-75)(200+75)}{300+3(75)}$$

$$= 13.09524$$

(r) $$\mathring{e}_x = E[T_x] = \frac{(100-x)(200+x)}{300+3x}$$

(s) $$\mathring{e}_0 = E[T_0] = 66.\overline{66}$$

(t) $$\mathring{e}_{75} = E[T_{75}] = 13.09524$$

(u) $$\mathring{e} \ldots = \int^{10} \ldots dt$$

(u) $\mathring{e}_{75:\overline{10|}} = \int_0^{10} {}_tp_{75}\, dt$

$${}_tp_x = \frac{10{,}000 - (x+t)^2}{10{,}000 - x^2} \Rightarrow {}_tp_{75} = \frac{4375 - 150t - t^2}{4375}$$

$$\int_0^{10} \frac{4375 - 150t - t^2}{4375}\, dt$$

$$= \left.\frac{4375t - 75t^2 - t^3/3}{4375}\right|_0^{10}$$

$$= 8.20952$$

(v) $${}_{u|t}q_x = {}_up_x \cdot {}_tq_{x+u}$$

$$= \frac{10000 - (x+u)^2}{10{,}000 - x^2} \cdot \frac{2(x+u)(t) + t^2}{10000 - (x+u)^2}$$

$$= \frac{2(x+u)t + t^2}{10{,}000 - x^2}$$

(w) $${}_{10|5}q_{75} = \frac{(2)(75+10)(5) + 5^2}{10000 - 75^2}$$

$$= 0.2$$

(5) $$S_0(x) = e$$

$$= e^{-\int_0^x \frac{2}{100-s}\, ds}$$

$$= e^{-\left[2(-\ln(100-s))\right]_0^x}$$

$$= e^{2\ln(100-x) - 2\ln(100)}$$

$$= e^{2\ln(100-x) - 2\ln(100)}$$

$$= e^{2\ln\left(\frac{100-x}{100}\right)} = \left(\frac{100-x}{100}\right)^2$$

$$F_0(x) = 1 - S_0(x) =$$

$$1 - \left(\frac{100-x}{100}\right)^2$$

$$_{10}p_{50} = \frac{S_0(60)}{S_0(50)} = \frac{\left(\frac{100-60}{100}\right)^2}{\left(\frac{100-50}{100}\right)^2}$$

$$= \left(\frac{40}{50}\right)^2 = 0.64$$

(6) (a) $p_{x+3} = 1 - q_{x+3}$

$$= 1 - 0.02 = 0.98$$

(b) $_2p_x = p_x \cdot p_{x+1}$

$$= (0.99)(0.985)$$

$$= 0.97515$$

(c) $_3p_{x+1} = {_2p_{x+1}} \cdot p_{x+3}$

$$\therefore {_2p_{x+1}} = \frac{_3p_{x+1}}{p_{x+3}}$$

$$= \frac{0.95}{0.98} = 0.96939$$

from part a

(d) $_3p_x = p_x \cdot {_2p_{x+1}} =$

$$(0.99)\left(\frac{0.95}{0.98}\right) =$$

$$0.95969$$

(e) $_{1|2}q_x = p_x - {_3p_x}$

$$= 0.99 - 0.95969$$

$$= 0.03031$$

(7.) (a) $_tp_x = e^{-\int_0^t \mu_{x+s}\,ds}$

$$_tp_x^* = e^{-\int_0^t 2\mu_{x+s}\,ds}$$

$$= e^{-2\int_0^t \mu_{x+s}\,ds}$$

$$= \left(e^{-\int_0^t \mu_{x+s}\,ds}\right)^2$$

$$= \left(_tp_x\right)^2$$

(b) & (c) Done in spreadsheet

(8) (a) This probability can be expressed as

$$_{2|2}q_{91} = {_2p_{91}} - {_4p_{91}}$$

$$= (p_{91})(p_{92}) - (p_{91} \cdot p_{92} \cdot p_{93} \cdot p_{94})$$

but $p_x = 1 - q_x$ so

$$= (1-0.25)(1-0.30) -$$

$$(1-0.25)(1-0.30)(1-0.40)(1-0.50)$$

$$= 0.3675$$

(b) $$e_{90}^{f} - e_{90}^{m} = \sum_{1}^{\infty} {}_tp_x^{f} - \sum_{1}^{\infty} {}_tp_x^{m}$$

$$= {}_1p_x^{f} + {}_2p_x^{f} + \cdots - \left[ {}_1p_x^{m} + {}_2p_x^{m} + \cdots \right]$$

$$= .9 + (.9)(.85) + (.9)(.85)(.8) + (.9)(.85)(.8)(.75) + (.9)(.85)(.8)(.75)(.7) + (.9)(.85)(.8)(.75)(.7)(.6)$$

$$- \left[ .8 + (.8)(.75) + (.8)(.75)(.7) + (.8)(.75)(.7)(.6) + (.8)(.75)(.7)(.6)(.5) + (.8)(.75)(.7)(.6)(.5)(.4) \right]$$

$$= 1.00168$$

(c) $$e_{91:\overline{3|}} = \sum_{t=1}^{3} {}_tp_{91}$$

$$= {}_1p_{91} + {}_2p_{91} + {}_3p_{91} =$$

$$= 0.85 + (0.85)(.8) + (.85)(.8)(.75)$$

$$= 2.04$$

(9) $$e_{40} = e_{40:\overline{20|}} + {}_{20}p_{40}\, e_{60}$$

$$= 18 + (1 - 0.2)(25)$$

$$= 18 + (1-0.2)(25)$$

$$= 38$$

$$e_{40} = {}_1p_{40} + {}_1p_{40}\, e_{41}$$

$$38 = (1-0.003) + (1-0.003)(e_{41})$$

$$\therefore e_{4} = \frac{38 - .997}{.997} = 37.11434$$